What is ... a Coulomb gas ?

ELECTROSTATIC

- 2 electric charges q_0, q at position $x_0, x \in \mathbb{R}^3$ have an energy of interaction $V = \frac{q_0 q \cdot c}{|x_0 - x|}$
  q_0 exerts the force $-\nabla_x V$ on q

- in dimension d, it is rather V solution of
  $-\Delta_x V_d = \delta(x - x_0)$ ←

$$\rightsquigarrow \quad V_d = \begin{cases} -c_1 |x - x_0| & d = 1 \\ -c_2 \ln|x - x_0| & d = 2 \\ \dfrac{c_d}{|x - x_0|^{d-2}} & d \geq 3 \end{cases}$$

STATISTICAL PHYSICS

- studies systems of many constituents in interaction
  give up detailed deterministic description
  only retains statistical properties

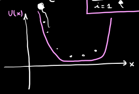
- $A \subseteq \mathbb{R}^d$ ambient space $\rightsquigarrow$ configuration space A^n
  probability measure $\mu \in \mathcal{P}(A^n)$
  $d\mu = \frac{1}{Z_n} \prod_{i=1}^n dx_i \, \boxed{e^{-\beta H}}$ Boltzmann law
  $H(x_1, \dots, x_n)$ energy of a configuration
  $\beta = 1/T$ inverse temperature

  $T \to 0 \iff \beta \to +\infty$ configurations minimizing H dominate

- System of electrons, ions (= plasma)
  $H_c = \sum_{i<j} V_d(x_i, x_j)$ (order n^2)
  $d \geqslant 2$: $x_i \to x_j \implies V_d(x_i, x_j) \longrightarrow +\infty$: repulsion

- Dynamic $\partial_t^2 x_i = -\underbrace{\nabla H} + \text{noise} + \text{external forces}$
  particles tend to escape at ∞

- In reality (stars, material, ...)
  $H = H_c + \boxed{\sum_{i=1}^{n} n \, U(x_i)}$
  - confinement $U(x) \xrightarrow[|x| \to \infty]{} +\infty$
  - electric field
  - mean effect of ambient material (nuclei in a crystal, ...)



We will restrict to study at equilibrium

THERMODYNAMIC LIMIT : $n \longrightarrow +\infty$

- configuration $(x_1, \dots, x_n) \in A^n$ $\longrightarrow$ empirical measure $L_n = \frac{1}{n} \sum_{i=1}^{n} \delta_{x_i} \in \mathcal{P}(A)$
  random n-tuple → random probability measure

$$d\mu(x_1,\dots,x_n) = \frac{\prod_{i=1}^n dx_i}{Z_n} \exp\Big[-\beta\Big(\underbrace{\sum_{i<j} V_d(x_i,x_j)}_{n^2} + n\underbrace{\sum_i U(x_i)}_{n}\Big)\Big]$$

$$= \frac{\prod_{i=1}^n dx_i}{Z_n} \exp\big[-\beta n^2 \underline{\mathcal{E}[L_n]}\big] \qquad n \to +\infty$$

energy functional $\mathcal{E} : \mathcal{P}(A) \to \mathbb{R} \cup \{\pm\infty\}$

$$\mathcal{E}[\lambda] = \frac{1}{2}\iint_{x\neq y} V_d(x,y)\,d\lambda(x)\,d\lambda(y) + \underline{\int U(x)\,d\lambda(x)}$$

$$= \frac{1}{2}\iint_{x\neq y}\big(V_d(x,y) + U(x) + U(y)\big)\,d\lambda(x)\,d\lambda(y)$$

- we expect L_n to be close to a minimizer of $\mathcal{E}$ when $n \to +\infty$

REMINDER on CONVERGENCE of RANDOM VARIABLES

- $(X_n : \Omega \to \mathbb{R})_{n\geq 1}$ sequence of r.v.

  $X : \Omega \to \underline{\mathbb{R}}$ $\mathcal{P}(A)$ r.v

  We say that X_n converges to X $n \to +\infty$

  - almost surely if $\mathbb{P}[\{\omega\in\Omega \mid \lim_{n\infty} X_n(\omega) = X(\omega)\}] = 1$
  - in expectation (L^1) if $\mathbb{E}[|X_n - X|] \xrightarrow[n\infty]{} 0$
  - in law if $\mathbb{P}[X_n \leq x] \xrightarrow[n\infty]{} \mathbb{P}[X \leq x]$ at every point of continuity $x \in \mathbb{R}$ of

  This uses the topology on $\mathbb{R}$.

- $\mathcal{P}(A)$ has several topologies.

  We choose the weak topology (makes it a Polish space)

  $\lambda_n \xrightarrow[n\infty]{w} \lambda$ if $\forall f \in C_b^0(A)$ $\int_A f\,d(\lambda_n - \lambda) \xrightarrow[n\infty]{} 0$

  A sequence $(\lambda_n)_{n\geq 1}$ of random proba. measures on A converges to a random proba. measure λ

  - almost surely if $\mathbb{P}[\{\omega\in\Omega \mid \lambda_n(\omega) \xrightarrow{w} \lambda(\omega)\}] = 1$
  - in expectation if $\forall f \in C_b^0(A)$ $\mathbb{E}\Big[\underline{\Big|\int_A f\,d(\lambda_n - \lambda)\Big|}\Big] \xrightarrow[n\infty]{} 0$

  ~~in law~~

EQUILIBRIUM MEASURE

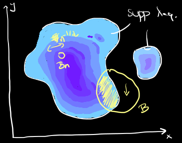
THEOREM — Assume U continuous, grows fast enough at ∞

① $\mathcal{E}$ admits a unique minimizer λ_{eq}

② λ_{eq} has compact support

③ $Z_n = \int_{A^n} \prod_{i=1}^n dx_i\, e^{-\beta H} = \exp[-\beta n^2 \mathcal{E}(\lambda_{eq}) + O(n \ln n)]$ partition function

④ $L_n \xrightarrow[L^1]{a.s} \underline{\lambda_{eq}}$



density plot of λ_{eq} $(d=2)$

⑤ $\boxed{\lambda_{eq}^{(n)}} \xrightarrow[n\infty]{W} \lambda_{eq}$ in $\mathcal{P}(A)$

↳ minimizer of $\mathcal{E}$ on the image of A^n in $\mathcal{P}(A)$ for finite n
[Fekete points]

TWO REGIMES OF STUDY

Macroscopic concerns the behavior of $O(n)$ particles

- law of large numbers $\frac{1}{n}\{\# x_i \in B\} \xrightarrow[n\infty]{} \int_B d\lambda_{eq}$
  $B \subseteq A$, independent of n

- fluctuations $\mathrm{Fluct}_n[f] = \left(\underbrace{\sum_{i=1}^{n} f(x_i)}_{} - n \underbrace{\int_A f \, d\lambda_{eq}}_{}\right)$
  $f \in C^0(A)$
  independent of n

  under extra assumptions - $\mathrm{Fluct}_n[f] = O(1)$

  → CLT - $\mathrm{Fluct}_n[f] \xrightarrow[n\infty]{law}$ Gaussian r.v
  mean $m[f]$
  covariance $Q[f,f]$

In fact, one expects (and sometimes prove)
that fluctuations have Gaussian free field behavior

Microscopic concerns the behavior of $O(1)$ particles

Heuristic : in $B_n \subset\subset \mathring{\mathrm{supp}}(\lambda_{eq})$ of diameter $O(n^{-1/d})$
there are typically $O(1)$ particles.

Sometimes [Sylvia's talk] local configurations in B_n
are disposed (with overwhelming probability) on a lattice

⇝ crystallization

IMPORTANT ? - Under which condition (d, β) ?
does crystallisation occur

- which lattice appears ?
(triangular, cubic, ...)

Macro : ~ easy, ppT governed by μ_{eq} (depend on V, not on β)
Micro : ~ hard, collective effects (depend on β, not on V).

Universality conjectures : microscopic ppT near $x_0 \in A$ only depend on
- local behavior of λ_{eq} at x_0
- type of short distance repulsion (β)

Coulomb gases are mathematical models to understand phases (crystallized or not) of matter

GAUSSIAN FREE FIELD

- Let $A \subseteq \mathbb{R}^d$ bounded, smooth ∂A
  $\mathcal{H} = \{ \psi \in L^2(A) \mid \psi_{|\partial A} = 0 \}$

  $-\Delta \psi_n = \underline{\omega_n} \boxed{\psi_n}$ $n \geq 1$
  Laplacian
  $0 < \omega_1 < \omega_2 < \cdots$

  ψ_n : modes of waves on A
  ω_n : "frequencies"

  Hilbert basis of $\mathcal{H}$

- Green function $-\Delta_x G(x, x_0) = \underline{\delta(x-x_0)}$

$\rightsquigarrow$ $G(x, x_0) = \sum_{n \geq 1} \omega_n^{-1} \psi_n(x_0) \psi_n(x)$

- The Gaussian free field $GFF(A)$
  the random ~~function~~ distribution. $\phi(x) = \sum' \boxed{a_n}\, \psi_n(x) \underline{\boxed{\omega_n^{-1/2}}}$
  $(a_n)_{n \geq 1}$ i.i.d Gaussian r.v
  mean 0, covariance 1

  $\rightarrow$ $\mathbb{E}[\phi(x)\phi(x_0)] = G(x, x_0)$

- It can be defined on any compact Riemannian (M, g) manifold (with boundary) $\underline{GFF(M,g)}$

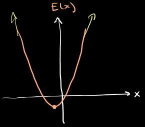
- For $d = 2$, under suitable assumption GFF behavior
  $\underbrace{\mathrm{Fluct}_n[f]}$ $\longrightarrow$ Gauß. covariance $Q(f) = \iint_{A^2} f(x) f(x_0) \underline{\tilde{G}(x, x_0)}$
  for $\tilde{G}$ = Green function of $(M, g) = (\underline{\mathrm{supp}\, \lambda_{eq}}, \boxed{\lambda_{eq}})$

EXISTENCE & UNIQUENESS OF λ_{eq}

- V grows fast at ∞ : can assume A bounded

  Recall $\mathcal{E}[\lambda] = \iint_{A^2} \underline{G(x,y)}\, d\lambda(x)\, d\lambda(y) + \int_A \underline{V(x)\, d\lambda(x)}$

  where G = Green function of $(\mathbb{R}^d, \text{euclidean})$



- $E : \mathbb{R}^d \to \mathbb{R}$ quadratic
  admits a unique minimum
  $\Leftrightarrow$ E is strictly convex

- More generally,
  if $E : \underset{\text{of } \mathbb{R}^d}{\text{convex domain}} \to \mathbb{R}$ is strictly convex and has compact level sets
  then $\exists!$ minimizer.

- Here

$$\mathcal{E}[(1-t)\lambda_0 + t\lambda_1] - (1-t)\,\mathcal{E}(\lambda_0) - t\,\mathcal{E}(\lambda_1)$$

$$= \iint_{A^2} \underbrace{G(x,y)} \underbrace{d(\lambda_1-\lambda_0)(x)\, d(\lambda_1-\lambda_0)(y)}$$

$\nu = \lambda_1 - \lambda_0$

$$= \sum_{n \geqslant 1}' \omega_n^{-1} \iint_{A^2} \psi_n(x)\,\psi_n(y)\, d\nu(x)\, d\nu(y)$$

$$= \sum_{n \geqslant 1}' \omega_n^{-1} \left| \int_A \psi_n(x)\, d\nu(x) \right|^2 = \infty^2(\nu).$$

$$\geqslant 0 \quad \text{with equality iff} \quad \underline{\nu = 0} \ \text{almost everywhere.}$$

Hence $\mathcal{E}$ strictly convex.

- Compact level sets come from $U \to +\infty$ fast enough at ∞
  (so λ in a level set cannot have mass going to ∞).

CHARACTERIZATION OF λ_{eq}

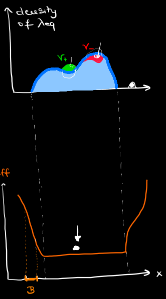
- $\forall$ signed measure ν on A of mass 0
  such that $\underline{\lambda_{eq}} + t\nu \geqslant 0$ for t small enough

$$0 \leq \mathcal{E}[\lambda_{eq} + t\nu] - \mathcal{E}(\lambda_{eq})$$

$$= \boxed{t} \underline{\int_A d\nu(x)\, V_{\text{eff}}(x)} + O(t^2)$$

with $V_{\text{eff}}(x) = \int_A d\lambda_{eq}(y)\, G(x,y) + \underline{U(x)}$

$$\Longrightarrow \begin{cases} V_{\text{eff}}(x) \geqslant c & \text{for some } c \in \mathbb{R} \\ V_{\text{eff}}(x) = c & \lambda_{eq} \text{ almost everywhere} \end{cases}$$



- V_{eff} is the effective (mean field) potential
  felt by a particle at $x \in A$

One can prove

$$\mathbb{P}[\exists i \ \ x_i \in B] = \exp\left(-n\beta \inf_{x\in B} V_{\text{eff}} + o(n)\right)$$

For $B \subset {}^c\mathrm{supp}\,\lambda_{eq}$ compact

$\rightsquigarrow$ large deviations of individual particles speed n

LARGE DEVIATIONS OF EMPIRICAL MEASURE

- use $a^2 - b^2 = (a-b)^2 + 2b(a-b)$

  with $a = L_n$, $b = \lambda_{eq}$

$$\mathcal{E}[L_n] - \mathcal{E}[\lambda_{eq}] = \left[\iint_{x\neq y} G(x,y)\, d(L_n-\lambda_{eq})(x)\, d(L_n-\lambda_{eq})(y)\right]$$

$$+ \int_A 2\, V_{\text{eff}}(x)\, d(L_n-\lambda_{eq})(x)$$

$$= \underbrace{\mathcal{D}^2(\lambda-\lambda_{eq})}_{} + 2\int_A \underbrace{(V_{\text{eff}}(x)-c)}_{\substack{0 \text{ on} \\ \mathrm{supp}\,\lambda_{eq}}}\underbrace{d(L_n-\lambda_{eq})(x)}_{\substack{\geqslant 0 \text{ on} \\ \mathrm{supp}\,\lambda_{eq}}} \Big) \geqslant 0.$$

$\geqslant 0$ on ${}^c\mathrm{supp}\,\lambda_{eq}$.

$$d\mu \propto \frac{1}{Z_n} e^{-\beta n^2 \mathcal{E}(L_n)} = \exp\left(\beta n^2(\mathcal{E}(\lambda_{eq}) - \mathcal{E}(L_n)) + \cdots\right).$$

so $\mathbb{P}[\underbrace{\mathcal{D}(L_n-\lambda_{eq})}_{} \geqslant t] \leq \exp(-\beta n^2 t^2 + O(n \ln n))$

- $\mathcal{D}$ controls linear statistics

$$\left|\int_A f\, d\nu\right| \leq \sum_{n\geqslant 1} |\langle f,\psi_n\rangle| \left|\int_A \psi_n\, d\nu\right| \leq \underbrace{\left(\sum_{n\geqslant 1} |\langle f,\psi_n\rangle|^2 \omega_n\right)^{1/2}}_{\|f\|_\Delta} \mathcal{D}(\nu)$$

$$\Longrightarrow \mathbb{P}\left[\left|\int f\, d(L_n-\lambda_{eq})(x)\right| \geqslant t\right] \leq \mathbb{P}\left[\mathcal{D}(L_n-\lambda_{eq}) \geqslant \frac{t}{\|f\|_\Delta}\right]$$

$$\leq \exp\left(-\frac{\beta n^2 t^2}{\|f\|_\Delta} + O(n\ln n)\right).$$

$$\boxed{\left|\int f\, d(L_n-\lambda_{eq})\right| \gg \underbrace{\sqrt{\frac{\ln n}{n}}\, \|f\|_\Delta^{1/2}}_{\varepsilon_n(f)} \text{ has probability} \to 0 \text{ as } n\to\infty}$$

large deviation estimate for Fluct_n

if f_n depends on n
only interesting when
$\varepsilon_n(f_n) \to 0 \Rightarrow$ macroscopic
mesoscopic
but not microscopic

This implies ④ $L_n \xrightarrow[L^1]{a.s} \lambda_{eq}$

⚠ Treatment of singularities in $G(x,y)$ require some care (regularisation).

UNIVERSALITY

1. " certain ppT of complicated systems may be the same as almost sure ppT in random ensembles of systems" (Wigner 50s)

2. same microscopic behavior if

$$\sum_{i<j}{}' G(x_i,x_j) \rightsquigarrow \sum_{i<j} R(x_i,x_j) \qquad R(x,x_0) \underset{x\to x_0}{\sim} G(x,x_0)$$

Coulomb   non Coulomb   but same singularity at short distance.

d = 2 IS SPECIAL

- $G(x,x_0) \sim -c \log|x-x_0| + O(1)$

  diverges both at short and long distance

  $\rightsquigarrow$ intricate both scales (technical difficulties)

  "log-correlated r.v"

Appear in

MATHS
- eigenvalues of random matrices
- zeroes of ζ

PHYSICS
- energy level of heavy nuclei