What is an ergodic decomposition of invariant measures?

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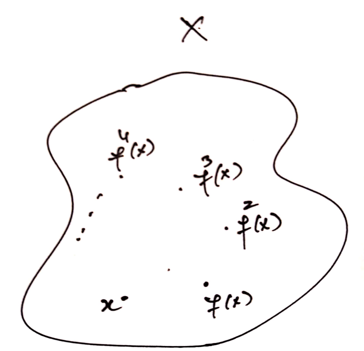
Set-up

- X denotes a complete separable metric space
- $\mathcal{A}$ denotes the Borel sigma-algebra, i.e, the the sigma-algebra generated by the open sets in X
- μ is a Borel probability measure
- $f : X \to X$ a measurable function



Ergodicity

Definition

The measure μ is said to be $f-$invaraint if for every measurable set $E \subseteq X$, we have $\mu(f^{-1}(E)) = \mu(E)$.

Example 1

The Lebesgue measure on $S^1 = R/Z$ is invariant under $f : S^1 \to S^2$ defined by $f(x) = x + \alpha$ (mod 1), for every real α.

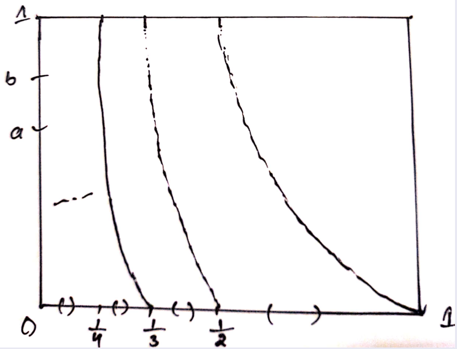
Example 2

Consider $X = [0, 1]$ and define

$$f(x) = \begin{cases} 0, & x = 0 \\ \frac{1}{x} \bmod 1, & x \in (0, 1], \end{cases}$$

$$\text{for } E \subseteq I, \mu(E) := \frac{1}{log2} \int_E \frac{1}{1+x} dx$$

Ergodicity



$$f^{-1}(a,b) = \bigcup_{n=1}^{\infty} \left(\frac{1}{n+b}, \frac{1}{n+a}\right)$$

Ergodicity

Fun Exercise

Compute all Lebesgue-volume preserving smooth maps $f : R^n \to R^n$ explicitly:

$$Det(\nabla f) = \pm 1.$$

Definition

The measure μ is Ergodic with respect to f if for every measurable set $E \subseteq X$ with $f^{-1}(E) = E \Rightarrow \mu(E) = 0$ or 1.

Example 1

$f : [0, 1] \to [0, 1]$ define by $f(x) = x^2$. The Dirac delta δ_0 is ergodic.

Theorem, Example 2

For each irrational α, the Lebesgue measure is the unique ergodic and invariant measure under $f : S^1 \to S^1$ defined by $f(x) = x + \alpha$ (mod 1).

Ergodicity

Theorem[Birkhoff, 1931]

Let μ be an f–invariant ergodic probability measure. For every $\phi \in L^1(\mu)$ we have

$$\lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1}\phi(f^i(x)) = \int_X \phi d\mu,$$

for μ-almost every $x \in X$.

Remark

Take $\phi = 1_E$, for a measurable set E, we have

$$\lim_{n\to\infty} \frac{1}{n}\sharp\{0 \leq i \leq n-1 : f^i(x) \in E\} = \lim_{n\to\infty} \frac{1}{n}\Sigma_{i=0}^{n-1} 1_E(f^i(x)) = \mu(E),$$

for μ-almost every $x \in X$.

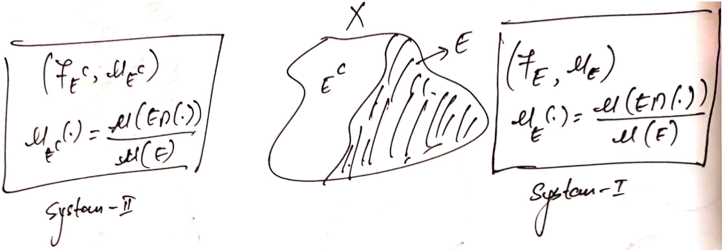
Motivation for Ergodic decomposition

Assume μ is not f-ergodic. Choose E such that $0 < \mu(E) < 1$ and $f^{-1}(E) = E$. Then $f^{-1}(E^c) = E^c$ and $0 < \mu(E^c) < 1$. So we can decompose (f, μ) into systems $(f|_E, \mu_E)$ and $(f|_E^c, \mu_{E^c})$, where

$$\mu_E(.) = \frac{\mu(E \cap (.))}{\mu(E)}$$

and similarly u_E^c is defined. Observe that

$$\mu = \mu(E)\mu_E + \mu(E^c)\mu_{E^c}.$$



Ergodic decomposition

Question 1

Given a partition $\mathbb{P}$ of X into measurable subsets: Is it possible to "disintegrate" μ into "conditional" measures on the elements of the partition $\mathbb{P}$?

Question 2

Assume (f, μ) is invariant but not ergodic. Does there exist a partition $\mathbb{P}$ which "disintegrate" μ into "conditional" measures $\{\mu_P : P \in \mathbb{P}\}$ such that each (f, μ_P) is invariant and ergodic?

Quotient Measure

Definition (Quotient sigma-algebra)

Let $(X, \mathcal{A}, \mu)$ be a Borel probability space and $\mathbb{P}$ be a partition of X into measure subsets. Consider the map $\tau : X \to \mathbb{P}$ $x \to P(x) \in \mathbb{P}$. Define the sigma-algebra $\mathcal{C}$ on $\mathbb{P}$ as

$$\text{For } A \subseteq \mathbb{P}, A \in \mathcal{C} \Leftrightarrow \tau^{-1}A \in \mathcal{A}.$$

Definition (Quotient Measure)

The quotient measure on $\mathbb{P}$ is the probability measure $\widehat{\mu} : \mathcal{C} \to [0, 1]$ defined by

$$\widehat{\mu}(A) = \mu(\tau^{-1}A) \text{ for all } A \in \mathcal{C}.$$

Definition (Decomposition of a Measure)

Let $(X, \mathcal{A}, \mu)$ be a probability space and let $\mathbb{P}$ be a partition of X. A family of probability measures $\{\mu_P : P \in \mathbb{P}\}$ on X is said to decompose μ w.r.t $\mathbb{P}$ if the following hold:

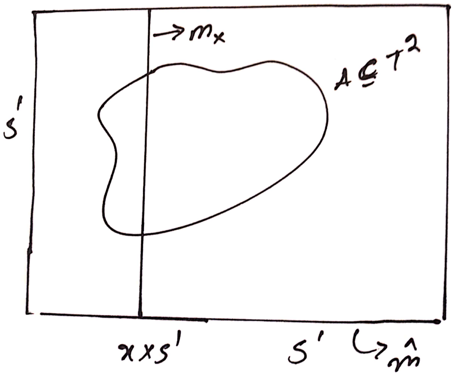
1. $\mu_P(P) = 1$ for $\widehat{\mu}$–almost every $P \in \mathbb{P}$.
2. For every measurable subset E of X, the map $P \to \mu_P(E)$ is measurable.
3. For every measurable subset E of X, $\mu(E) = \int_{\mathbb{P}} \mu_P(E)\, d\widehat{\mu}(P)$.

Definition (Ergodic Decomposition)

Let μ be f–invariant. An ergodic decomposition of of μ w.r.t the partition $\mathbb{P}$ is a decomposition of μ into probability measures $\{\mu_P : P \in \mathbb{P}\}$ on X, where $\hat{\mu}$–almost every μ_P is invariant and ergodic.

Example for existence of decomposition

$T^2 = S^1 \times S^1$, endowed with the Lebesgue measure m and take $\mathbb{P} = \{x \times S^1 : x \in S^1\}$.



Example for existence of decomposition

Let $\widehat{m}$ be the Lebesgue measure on S^1 and m_x be the Lebesgue measure on the fiber $x \times S^1$ measuring arc length. By the Fubini's theorem we have

$$\begin{aligned} m(A) &= \int_{S^1 \times S^1} \chi_A \, d(\widehat{m} \times \widehat{m}) \\ &= \int_{S^1} \Big(\int_{S^1} \chi_A \, d\widehat{m}(y) \Big) \, d\widehat{m}(x) \\ &= \int_{S^1} m_x(A) \, d\widehat{m}(x), \end{aligned}$$

for every measurable set E. This proves that $\{m_x : x \in S^1\}$ disintegrates m w.r.t $\mathbb{P}$.

Example for non-existence of decomposition

Let $X = S^1$ with Lebesgue measure m. Let $f : S^1 \to S^1$ be an irrational rotation.

claim

W.r.t the partition into orbits by f, $\mathbb{P} = \left\{ P_x = \{f^n(x)\}_{n \in Z} : x \in S^1 \right\}$, there does not exist any disintegration of m .

We prove that if $\left\{ \mu_{P_x} : P_x \in \mathbb{P} \right\}$ is a decomposition, then the family of the pull-backs

$$\left\{ f_\star \mu_{P_x}(.) = \mu_{P_x}(f^{-1}(.)) : P_x \in \mathbb{P} \right\}$$

is a also a decomposition of m.

Example for non-existence of decomposition

Theorem (Uniqueness of decomposition)

Let (Y, C, μ) be a probability space and suppose that the sigma-algebra C has a countable generator. If $\{\mu_P : P \in \mathbb{P}\}$ and $\{\acute{\mu}_P : P \in \mathbb{P}\}$ are two disintegrations of μ w.r.t a partition $\mathbb{P}$ of Y , then $\acute{\mu}_P = \mu_P$ for $\widehat{\mu}$-almost every $P \in \mathbb{P}$.

Proof of the claim

- $\mu_{P_x}(P_x) = 1$ for $\widehat{\mu}$–almost every $P_x \in \mathbb{P}$ implies $f_\star \mu_{P_x}(P_x) = \mu_{P_x}(f^{-1}(P_x)) = \mu_{P_x}(P_x) = 1$ for $\widehat{\mu}$–almost every $P_x \in \mathbb{P}$.

- For every measurable subset E of Y, the map $P_x \to \mu_{P_x}(E)$ is measurable implies $P_x \to f_\star \mu_{P_x}(E)$ is measurable.

Example for non-existence of decomposition

- By the invariance of m we have

$$\int_{\mathbb{P}} f_\star \mu_{P_x}(E)\, d\widehat{\mu}(P) = \int_{\mathbb{P}} \mu_{P_x}(f^{-1}(E))\, d\widehat{\mu}(P) = m(f^{-1}(E)) = m(E)$$

So $\{f_\star \mu_{P_x} : P_x \in \mathbb{P}\}$ is also disintegration of m w.r.t $\mathbb{P}$. By the uniqueness theorem above, we have $f_\star \mu_{P_x} = \mu_{P_x}$ for $\widehat{\mu}$–almost every $P_x \in \mathbb{P}$. Thus μ_{P_x} is f-invariant for $\widehat{\mu}$–almost every $P_x \in \mathbb{P}$.

The Lebesgue measure m is the only invariant measure $\Rightarrow \mu_{P_x} = m$ for $\widehat{\mu}$–almost every $P_x \in \mathbb{P}$ but $m(P_x) = 0$ and $\mu_{P_x}(P_x) = 1$. $\Rightarrow\Leftarrow$

So $\mathbb{P} = \{P_x = \{f^n(x)\}_{n\in Z} : x \in S^1\}$ does not disintegrate m. Q.E.D

Example for ergodic decomposition

Take $X = [0,1]$ and define $f : X \to X$ by $f(x) = x^2$. Consider the partition

$$\mathbb{P} = \{P_1 = \{0\}, P_2 = \{1\}, P_3 = (0,1)\}.$$

Claim: $\{\mu_{P_1} = \delta_0, \mu_{P_2} = \delta_1\}$ is the ergodic decomposition of every $f-$invariant Borel probability measure μ on X.

Proof of the claim

(1) For $0 < \epsilon < 1$, by invariance we have $\mu([0,\epsilon^n]) = \mu([0,\epsilon])$ for all $n \in N$.

(2) By the continuity of μ we have $\mu(\{0\}) = \mu([0,\epsilon])$ which means $\mu((0,\epsilon]) = 0$.

(3) Take $\epsilon = 1 - 1/2n$, we have $\mu((0, 1 - 1/2n]) = 0$ for all $n \in N$. By the continuity of μ we get $\mu((0,1)) = 0$.

Proof of the claim

(4) Therefor $\mu(\{0\},\{1\}) = 1$. Since $\widehat{\mu}(0,1) = 0$, the family $\{\mu_{P_1} = \delta_0, \mu_{P_2} = \delta_1\}$ disintegrates every invariant Borel probability measure μ,

(5) i.e, $\mu = \mu\{0\}\delta_0 + \mu\{1\}\delta_1$. Q.E.D

Question

Under which conditions a partition $\mathbb{P}$ decomposes a given measure μ?

V. A. Rohlin, 1947

On the Fundamental Ideas of Measure Theory, American Mathematical Society, 1952.

Measurable Partitions

Definition (refinement)

Let $\mathbb{P}_1, \mathbb{P}_2$ be two partitions of a set Y. We say $\mathbb{P}_2$ is a refinement of $\mathbb{P}_1$ if each $P \in \mathbb{P}_1$ is a union of elements of $Q \in \mathbb{P}_2$. We write $\mathbb{P}_1 \preceq \mathbb{P}_2$ when $\mathbb{P}_2$ is a refinement of $\mathbb{P}_1$.

Definition (Measurable Partitions)

Let $(Y, \mathcal{C}, \mu)$ be a probability space. A partition $\mathbb{P}$ of Y is measurable w.r.t μ if and only if $\exists$ a set $Y_0 \subseteq Y$ with $\mu(Y_0) = 1$ and a sequence of countable partitions $\mathbb{P}_n$, each consisting of measurable sets, such that $\mathbb{P}_n \preceq \mathbb{P}_{n+1}$ for all $n \in N$ and

$$\mathbb{P}\,|_{Y_0} = \Big\{ \bigcap_{n \in N} P_n : P_n \in \mathbb{P}_n \forall\, n \in N \Big\}.$$

Rokhlin disintegration theorem

Rokhlin disintegration theorem

Let $(X, \mathcal{A}, \mu)$ be a Borel probability space and $\mathbb{P}$ be a partition of X. There exists a disintegration of μ with respect to $\mathbb{P}$ if $\mathbb{P}$ measurable with respect to μ.

For $P \in \mathbb{P}$, choose a sequence $P_n \in \mathbb{P}_n$ such $P = \bigcap_{n \in N} P_n$. The limit

$$\mu_P(.) = \lim_{n \to \infty} \frac{\mu(P_n \cap (.))}{\mu(P_n)}.$$

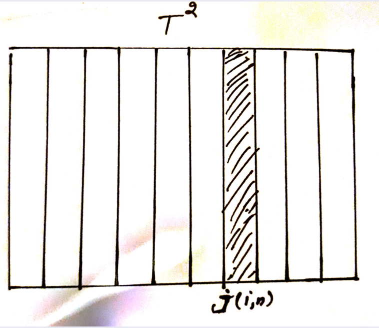
exists and is defined to be the conditional measure μ_P.

Example of measurable partition

$X = S^1 \times S^1$ with Lebesgue measure

$$\mathbb{P} = \{x \times S^1 : x \in S^1\}$$

$$\mathbb{P}_n = \{J(i,n) \times S^1 : i = 1, 2, 3, \ldots, 2^n\}, \text{ where } J(i,n) = [\frac{i-1}{2^n}, \frac{i}{2^n}).$$



A non-measurable partition

Definition

Let M be a compact Riemannian manifold. A diffeomorphism $f : M \to M$ is called Anosov diffeomorphism if there exist $c > 0$ and $0 < \lambda < 1$ such that for every $z \in M$,

- $T_z M = E_z^s \bigoplus E_z^u$,
- $Df(z)(E_z^s) = E_{f(z)}^s$ and $Df(z)(E_z^u) = E_{f(z)}^u$,
- $\|Df(z)^n(v)\| \leq c\lambda^n \|v\| \;\; \forall\, v \in E_z^s$,
- $\|Df(z)^{-n}(v)\|\| \leq c\lambda^n \|v\| \; \forall\, v \in E_z^u$.

Theorem, D. Anosov

Every C^2 measure-preserving Anosov diffeomorphism is ergodic.

A non-measurable partition

Open Problem 1

Is the theorem true for C^1 Anosov diffeomorphism?

Open Problem 2

Classify manifolds that admits Anosov diffeomorphism and that don't....

A non-measurable partition

$T^2 = R^2/Z^2$ with Lebesgue measure m and the Anosov diffeomorphsim $f : T^2 \to T^2$ defined by

$$\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \text{mod } 1.$$

The $\mathbb{P} = \{\mathcal{W}^u(x) : x \in T^2\}$ into the unstable manifolds is not measurable w.r.t m.

Proof sketch

- Assume $\mathbb{P}$ is measurable w.r.t m. Take $\mathcal{W}^u(x) \in \mathbb{P}$, there exists a sequence $P_n \in \mathbb{P}_n$ such that $\mathcal{W}^u(x) = \cap_{n \in N} P_n$.
- $P_n \in \mathbb{P}_n$ is a union of unstable manifolds and hence invariant under f, i.e, $f^{-1}(P_n) = P_n$.
- m is invariant under f and f is C^2, so by the above theorem of Anosov m is ergodic. By ergodicity either $m(P_n) = 1$ or 0. But $\mathbb{P}_n$ is a partition of T, so for each $n \in N$, there exists $P_n \in \mathbb{P}_n$ such that $\mu(P_n) = 1$
- $\mathcal{W}^u(x) = \cap_{n \in N} P_n$ has full lebesgue measure which is absurd since $W^u(x)$ a one dimensional object ($\mathbb{P}$ is a 1-dimensional foliation).

Ergodic decomposition theorem

Question 2

Assume (f, μ) is invariant but not ergodic. Does there exist a partition $\mathbb{P}$ which "disintegrate" μ into "conditional" measures $\{\mu_P : P \in \mathbb{P}\}$ such that each (f, μ_P) is invariant and ergodic?

Theorem

Let $f : X \to X$ be a measurable map and μ be an f-invariant probably measure. The following are equivalent:

- μ is ergodic.
- For every measurable set $A \subseteq X$ the function $\tau(., A) : X \to R$ defined by
$$\tau(x, A) = \lim_{n\to\infty} \frac{1}{n} \sharp\{0 \leq i \leq n-1 : f^i(x) \in A\}$$
is constant at μ-almost every point of X.

Ergodic decomposition theorem

Definition (Dynamical Partition)

Let $(X, \widehat{\mathcal{A}}, \mu)$ be a Borel probability space, and and $f : X \to X$ a measurable transformation. Define an equivalence relation on X as

$$x \backsim y \Leftrightarrow \tau(x, A) = \tau(y, A) \text{ for every } A \in \mathcal{A}.$$

The dynamical partition $\mathbb{P}_f$ of X with respect to f is the partition into equivalence classes defined by the equivalence relation above.

Ergodic decomposition theorem

The dynamical Partition $\mathbb{P}_f$ above is measurable w.r.t μ and the Rokhlin disintegration $\{\mu_P : P \in \mathbb{P}_f\}$ of μ is the ergodic decomposition of μ if μ is f-invariant.

Computing the ergodic decomposition

Example

Take $X = S^1 \times S^1$ with the Lebesgue measure m. Define $f : \mathbb{T}^2 \to \mathbb{T}^2$ by

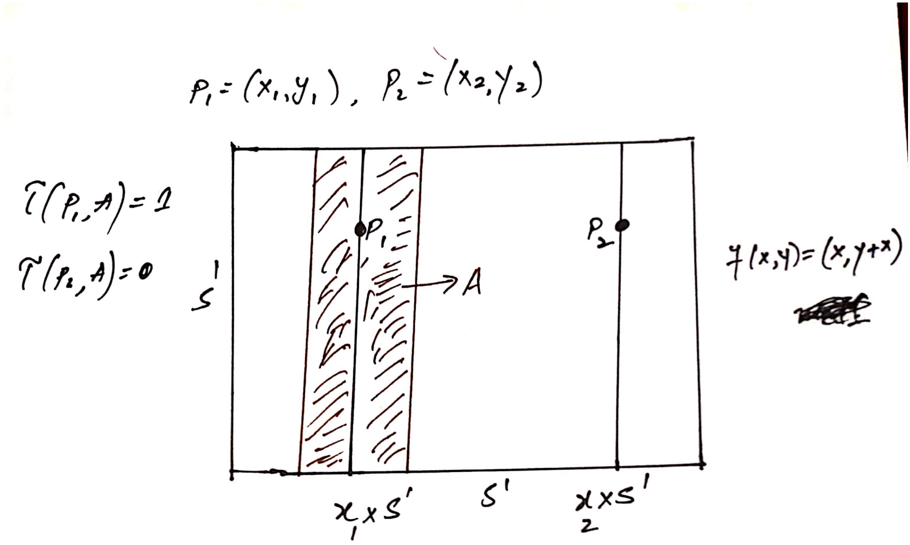
$$f(x, y) = (x, y + x) \text{ mod } 1 \ .$$

claim

The dynamical Partition $\mathbb{P}_f$ of $\mathbb{T}^2$ is the vertical fibers, i.e,

$$\mathbb{P} = \{x \times S^1 : x \in S^1\}.$$

Computing ergodic decomposition



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