What is ... similarity of shapes?

Daria Schymura
FU Berlin

19 December 2008

Shape Matching

Given two objects, how much do they resemble each other?

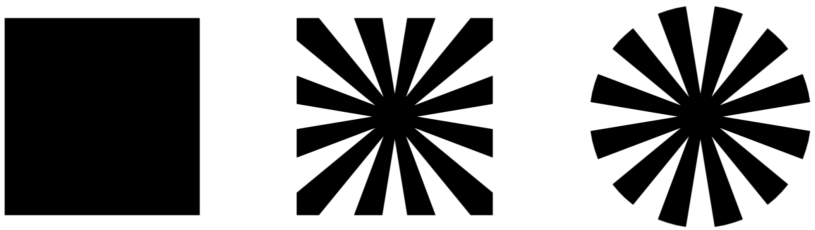
- many different motivations/applications
  - computer vision
  - object recognition, special case: character recognition
  - image retrieval
  - drug design/molecular docking
  - registration of medical images
- many different problem types
- many different fields in computer science, mostly heuristics
- here: computational geometry, aims for exact, provably correct and efficient solutions

Problems from the Application Point of View

Human Perception is not a metric:

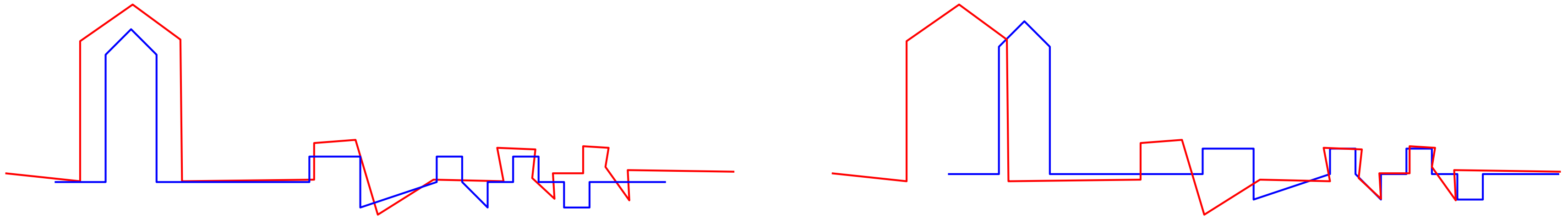
symmetry

triangle inequality



Problems from the Application Point of View

Partial matching:
size of matched part vs. quality of the matching



Mathematical Point of View

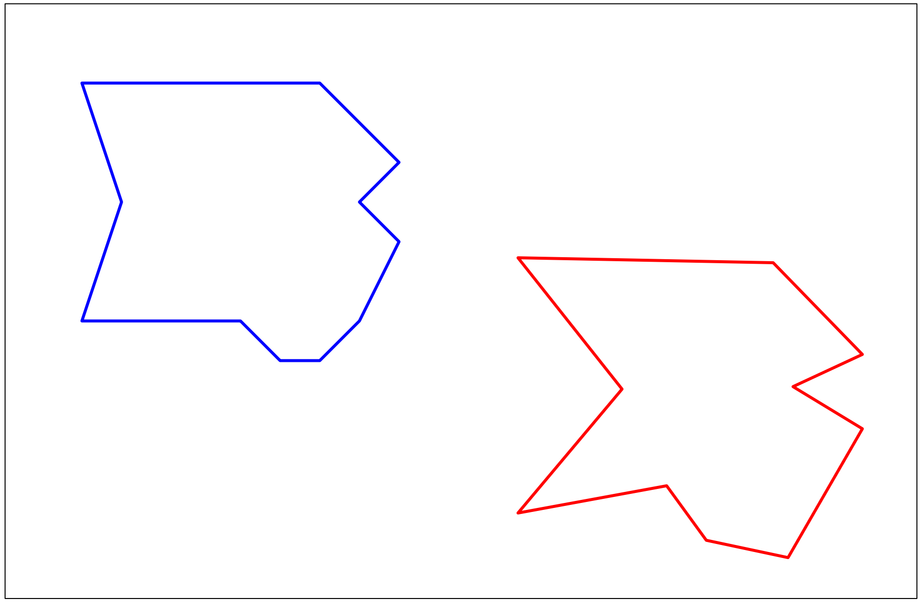
General Matching Problem

Given shapes A, B, transformation group T, distance measure d.

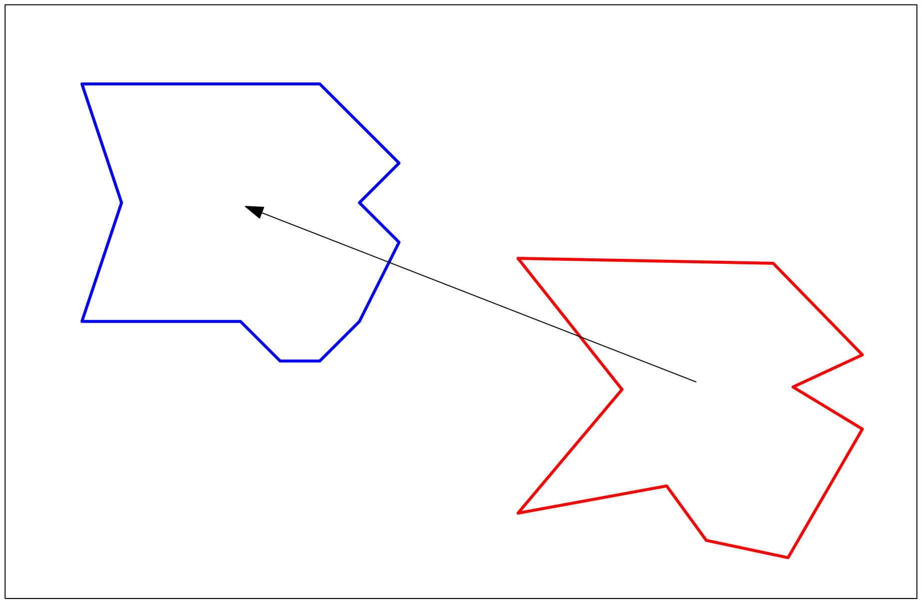
Optimization Problem
Compute $t \in T$ such that $d(t(A), B)$ is minimal.

Decision Problem
For a threshold $\varepsilon \geq 0$, is there a $t \in T$ such that $d(t(A), B) < \varepsilon$?

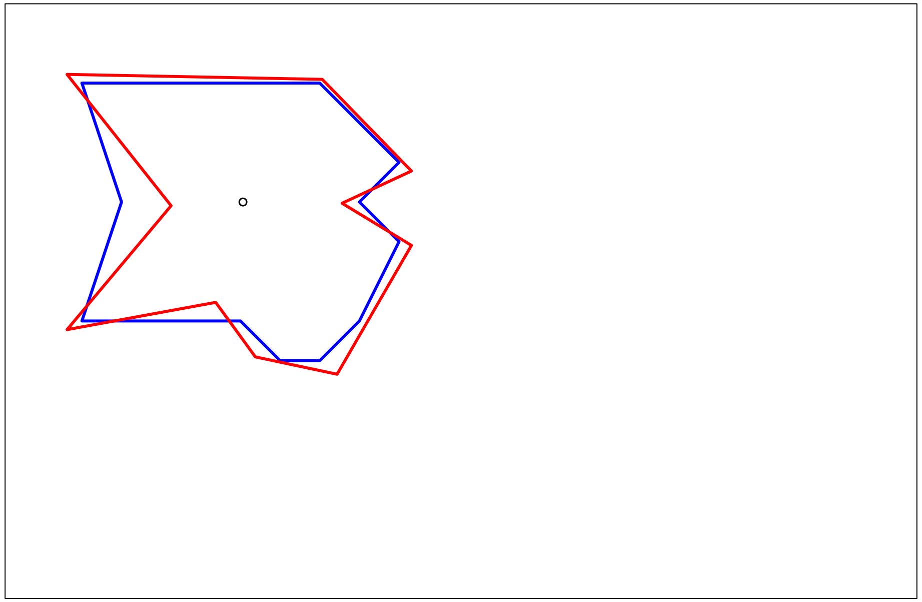
Example



Example



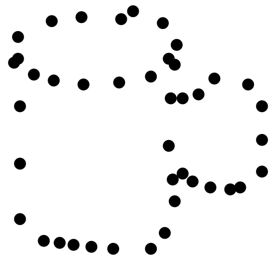
Example



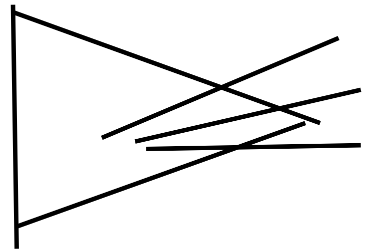
Shapes

Mostly in $\mathbb{R}^2$, sometimes $\mathbb{R}^3$ or $\mathbb{R}^d$.

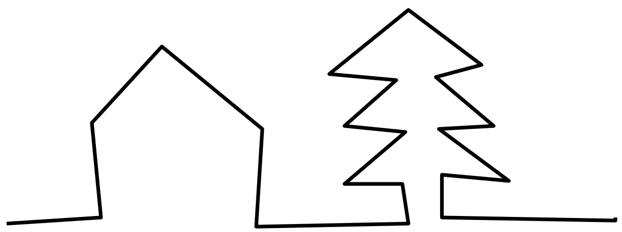
discrete point set



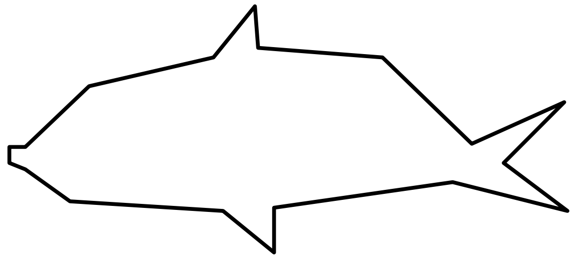
set of line segments



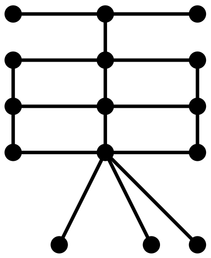
polylines



polygons



geometric graphs



Transformation Groups

- translations
- homotheties (translation + scaling)
- rigid motions (translation + rotation)
- similarities (translation + rotation + scaling)
- affine transformations

Distance Measures

- discrete metric
- Bottleneck distance
- (directed) Hausdorff distance
- (discrete) Fréchet distance
- area of symmetric difference
- turning function
- Earth Movers Distance

Discrete Metric

$$d(A,B) = \begin{cases} 0 & \text{if } A = B \\ 1 & \text{otherwise} \end{cases}$$

Discrete point sets can be matched in $O(n \log n)$ time under translations and rigid motions.

Big-O-Notation

$f : \mathbb{N} \to \mathbb{N}$ running time of an algorithm
(in number of elementary operations).

$g : \mathbb{R} \to \mathbb{R}^+$ a function.

$$f(n) = O(g(n))$$

if there is a constant $C > 0$ and $N \in \mathbb{N}$ such that for all $n > N$

$$f(n) \leq C\ g(n).$$

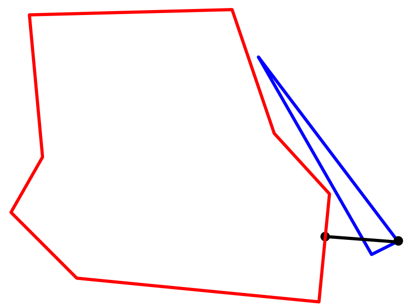
Usually, "efficient" means polynomial.

Hausdorff distance

A, B compact sets in $\mathbb{R}^d$

directed Hausdorff distance

$$\vec{d}_H(A,B) = \max_{a\in A}\min_{b\in B}\|a-b\| = \max_{a\in A}\mathrm{dist}(a,B)$$



- depends on a metric on $\mathbb{R}^d$, for example L_1, L_2 or L_∞.
- allows partial-complete matching
- distance is determined by worst point pair

Hausdorff distance

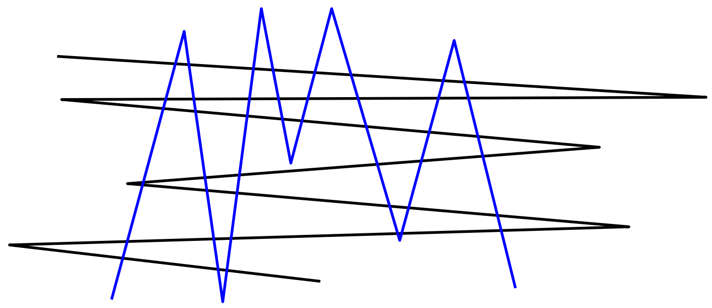
undirected Hausdorff distance

$$d_H(A,B) = \max\{\vec{d}_H(A,B), \vec{d}_H(B,A)\}$$

- very common distance measure
- efficient algorithms for computation and matching of discrete point sets and sets of line segments under translations and rigid motions

Fréchet distance

For curves, the Hausdorff distance is not appropriate:



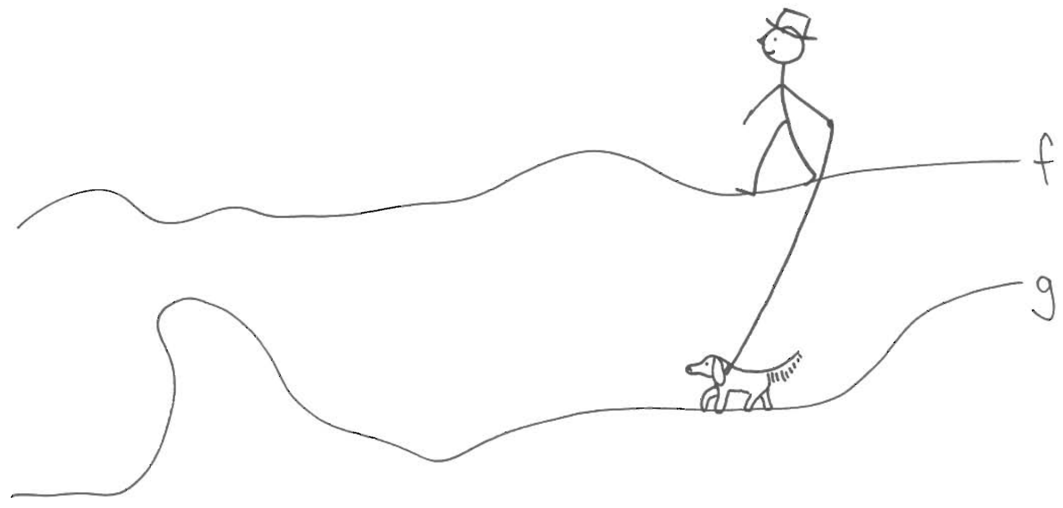
Fréchet distance (1906) of two curves $f, g : [0,1] \longrightarrow \mathbb{R}^d$

$$d_F(f,g) = \inf_{\alpha,\beta} \max_{t\in[0,1]} \|f(\alpha(t)), g(\beta(t))\|$$

$\alpha, \beta : [0,1] \rightarrow [0,1]$ continuous, monotone increasing functions

Fréchet distance

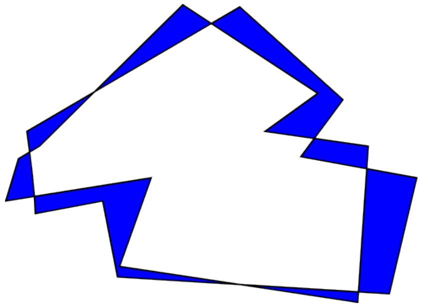
Man-Dog-distance / leash distance



- Efficient algorithms for computation and matching under translation ($O(n^8 \log n)$) are known for polygonal curves.
- There is a discrete version of Fréchet distance.
- Higher-dimensional Fréchet distance seems to be very hard.

Area of Symmetric Difference

Natural distance for polygons.



- Matching of polygons with n vertices under translations in $O(n^4)$ time.
- Faster probabilistic approximation algorithms.
- convex polygons: $O(n \log n)$
- For rigid motions, there is a probabilistic algorithm that computes an absolute error approximation in $O(n^3/\varepsilon^4 \log^5 n)$.

Earth Movers Distance

Monge-Kantorovich mass transportation problem (1781,1942)

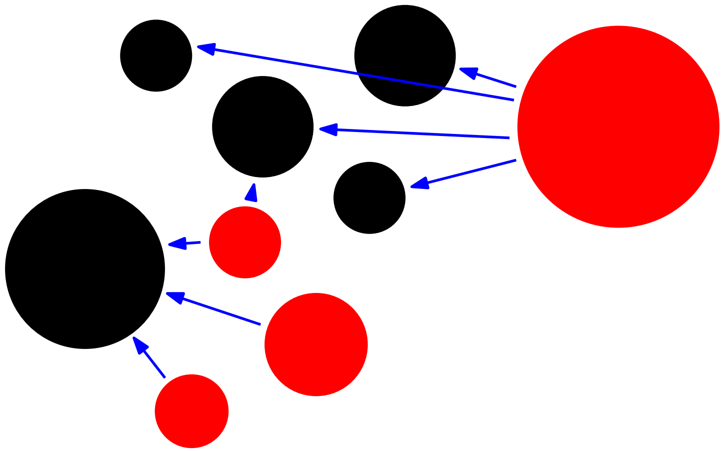
Mass distributions P, Q on $(E, \mathcal{A})$, cost function $c : E \times E \longrightarrow \mathbb{R}^+$.

Determine an optimal transportation plan μ^* on the class of probability measures on $(E \times E, \mathcal{A} \times \mathcal{A})$ with marginals P, Q such that the transportation cost is minimal.

$$d_T(P, Q) = \inf_{\mu} \int c(x, y) d\mu(x, y)$$

Earth Movers Distance

Given a set of piles of earth and a set of holes in the ground, what is the minimal amount of work needed to fill the holes with earth?



- metric on weighted point sets with equal total weight in $\mathbb{R}^d$
- can be formulated as uncapacited minimum cost flow problem
- efficient algorithms for computation, approximation algorithms for matching under translation and rigid motions