What Is... Brownian Motion?

February 17, 2009

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- A *filtration* $(\mathscr{F}_t)_{t \geq 0}$ is a set of sub-σ-algebras of the set of events, with the property $\mathscr{F}_t \subseteq \mathscr{F}_s \quad \forall s \geq t, t \geq 0$. $\mathscr{F}_t$ represents the *information* available to an observer after time t.

Normal distribution

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A random variable $X : \Omega \to \mathbb{R}$ is said to have a normal distribution with mean μ and variance σ if

$$\mathbb{P}(X < x) = \int_{-\infty}^{x} \frac{1}{\sqrt{2\pi\sigma^2}} \exp\Big(- \frac{(z-\mu)^2}{2\sigma^2} \Big) \mathrm{d}z$$

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1. $B_0 = 0$ a.s.;
2. for $0 \leq s < t$, the increment $B_t - B_s$ is independent of $\mathscr{F}_s$. In particular, this means that, for any $0 \leq t_1, \ldots, t_n < \infty$, the *increments* $B_{t_2} - B_{t_1}, \ \ldots \ , B_{t_n} - B_{t_{n-1}}$ are independent;

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3. for $0 \leq s < t$, the increment $B_t - B_s$ is normally distributed with mean 0 and variance $t - s$.

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Quadratic Varience

$$< B >_t = \lim_{M(D)\downarrow 0} \sum_D |B_{t_i} - B_{t_{i-1}}|^2 = t$$

Properties

Nowhere differentiability

For almost every $\omega \in \Omega$ the Brownian path $B_t(\omega)$ is nowhere differentiable.

Distributional Invariance

Scaling

For $c > 0$, $\{X_t, \mathscr{F}_{ct}; 0 \leq t < \infty\}$ where $X_t = \frac{1}{\sqrt{c}} B_{ct}$ is a standard Brownian Motion.

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Lévy Characterization

Let M be a continuous Martingale process with initial zero, with bracket process $< M >_t = t$. Then M is a Brownian Motion.

Proof of existence of Brownian Motion.

However, it is by no means obvious that a process with the properties defined above exists.

The proof is based in the Wiener construction of Brownian Motion, and it has been modified and simplified by Lévy and Cielielski.

Proof of existence of Brownian Motion.

Construction...

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Let $\{\xi_k^{(n)}; n \in \mathbb{N}, k \in I(n)\}$ be a countable set of independent random variables on the probability triple $(\Omega, \mathscr{F}, \mathbb{P})$ with

- $\xi_k^{(n)}$ have a normal distribution with mean zero and variance 1.
- $I(n)$ is the set of all odd natural numbers that are smaller than 2^n.

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Given $n \in \mathbb{N}$, suppose that the value $B_{k/2^{n-1}}^{(n-1)}$ have been specified for $k = 0, 1, ..., 2^{n-1}$. We define $\{B_t^{(n-1)}; 0 \leq t \leq 1\}$ by interpolation of the specified points.

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Define $s = (k-1)/2^n$, $t = (k+1)/2^n$, $\mu = (B_t^{(n-1)} + B_s^{(n-1)})/2$, and $\sigma^2 = (t-s)/4 = 1/2^{n+1}$.

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$$B_{(t+s)/2}^{(n)} := \mu + \sigma \xi_k^{(n)}$$

for all $k = 1, ..., 2^{n-1} - 1$.

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for all $k = 1, ..., 2^{n-1} - 1$. The full process $\{B_t^{(n)}; 0 \leq t \leq 1\}$ is similarly defined by interpolation.

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\begin{aligned}
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H_k^{(n)} &:= \begin{cases} 2^{(n-1)/2}, & \frac{k-1}{2^n} \leq t < \frac{k}{2^n} \\ -2^{(n-1)/2}, & \frac{k}{2^n} \leq t < \frac{k+1}{2^n} \\ 0, & \text{otherwise.} \end{cases}
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$S_k^{(n)}$ are *tent functions* height $2^{-(n+1)/2}$ centre $k/2^n$ and they are non-overlapping for $k \in I(n)$.

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From induction, we get

$$B_t^{(n)}(\omega) = \sum_{m=0}^{n} \sum_{k \in I(m)} \xi_k^{(m)}(\omega) S_k^{(m)}(t), \;\; 0 \leq t \leq 1; \;\; n \geq 0$$

Lemma.

As $n \to \infty$, the sequence of functions $\{B_t^{(n)}(\omega); 0 \leq t \leq 1\}$ converges *uniformly* in t to a continuous function $\{B_t(\omega); 0 \leq t \leq 1\}$ a.s. in Ω.

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&= \sqrt{\frac{2}{\pi}} \frac{e^{-x^2/2}}{x}
\end{aligned}$$

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$$\begin{aligned} \mathbb{P}(b_n > n) &= \mathbb{P}\Big(\bigcup_{k \in I(n)} \{|\xi_k^{(n)}| > n\} \Big) \\ &\leq 2^n \mathbb{P}(|\xi_1^{(n)}| > n) \end{aligned}$$

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We use the Borel-Cantelli Lemma to show that there is a set $\tilde{\Omega} \subseteq \Omega$ with $\mathbb{P}(\tilde{\Omega}) = 1$ on which there is random integer $n(\omega)$ such that $b_n(\omega) \leq n$ for all $n \geq n(\omega)$.

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$$\sum_{m=n(\omega)}^{\infty} \sum_{k \in I(m)} |\xi_k^{(m)}(\omega) S_k^{(m)}(t)| \leq \sum_{n=n(\omega)}^{\infty} n 2^{-(n+1)/2} < \infty$$

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In otherwords, $B_t^{(n)}(\omega)$ converges uniformly in t to a limit $B_t(\omega)$ whenever $\omega \in \tilde{\omega}$. A basic result in analysis gives the conclusion that B is also continuous on $t \in [0, 1]$.

Theorem

The sequence $\{B_t^{(n)}; 0 \leq t \leq 1\}_{n=1}^{\infty}$ converges a.s. to a continuous process B_t, and the process $\{B_t, \mathscr{F}_t^B; 0 \leq t \leq 1\}$ is a Brownian Motion on $[0, 1]$.

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1. Show that the increments $\{B^{(n)}_{k/2^n} - B^{(n)}_{(k-1)/2^n}\}_{k=1}^{2^n}$ are independent, normal distributions with mean zero and variance 2^{-n}.
2. Let $0 = t_0 < t_1 < \ldots < t_n \leq 1$ be dyadic rational numbers. Show that the increments $\{B_{t_k} - B_{t_{k-1}}\}_{k=1}^{n}$ are independent, normal random variables with mean zero and variance $t_j - t_{j-1}$, respectively.

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3. Let $0 = t_0 < t_1 < \ldots < t_n \leq 1$ be real numbers. Show that the increments $\{B_{t_k} - B_{t_{k-1}}\}_{k=1}^{n}$ are independent, normal random variables with mean zero and variance $t_j - t_{j-1}$, respectively.