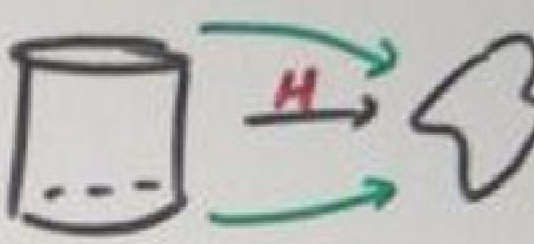
Basic Notions From Topology

Homotopy: Maps $f, g : X \to Y$ are **homotopic** if there exists a map $H : X \times I \to Y$

$$\begin{tikzcd}
X \arrow[d, "i_0"'] \arrow[dr, "f"] & \\
X \times I \arrow[r, "H"] & Y \\
X \arrow[u, "i_1"] \arrow[ur, "g"'] &
\end{tikzcd}$$



Homotopy equivalence: A map $f : X \to Y$ is a **homotopy equiv** if there exists $g : Y \to X$ s.t. $fg \simeq id_Y$ and $gf \simeq id_X$

Homotopy groups $\pi_n(X)$: Homotopy classes of (based) maps $S^n \to X$

weak homotopy equivalence: A map $f : X \to Y$ is a **weak h.e.** if $f_* : \pi_n(X) \to \pi_n(Y)$ is a bijection of pointed sets for $n = 0$ and an iso of groups for $n \geq 1$.

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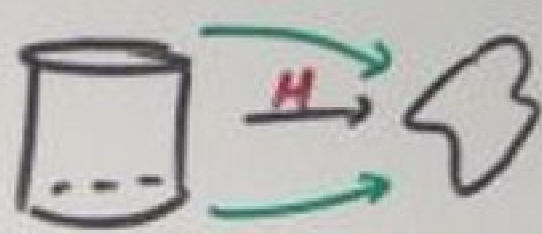

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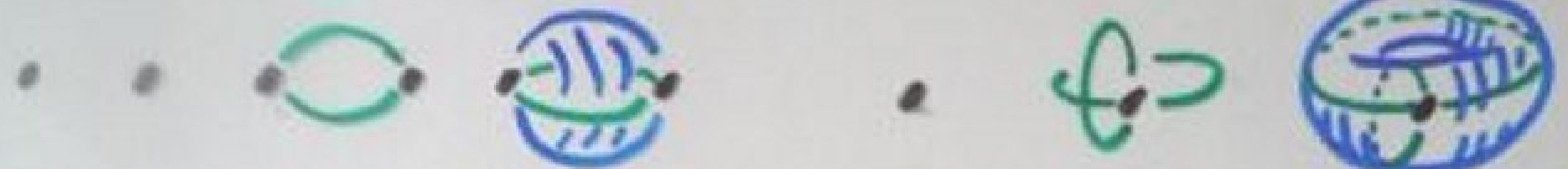
Weak homotopy equivalence: A map $f : X \to Y$ is a **weak h.e.** if $f_* : \pi_n(X) \to \pi_n(Y)$ is a bijection of pointed sets for $n = 0$ and an iso of groups for $n \geq 1$.

CW complexes: Spaces built inductively by "attaching cells"



Defn A map $f: X \to Y$ is a Serre fibration if for any CW complex A and any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "i_0"'] & X \arrow[d, "f"] \\ A \times I \arrow[r] \arrow[ur, dashed, "\ell"] & Y \end{tikzcd}$$

a lift ℓ exists.

(e.g. projections of products, covering spaces)

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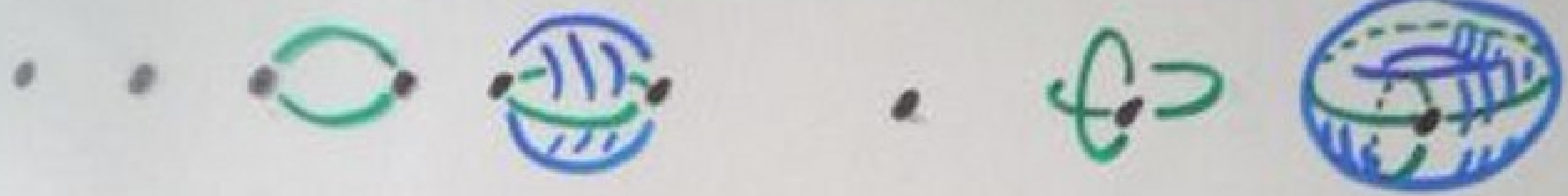

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Lemma A map $f: X \to Y$ is a S. fibration and a weak homotopy equivalence $\Leftrightarrow$ for any $i_n: S^{n-1} \hookrightarrow D^n$ and any diagram

$$\begin{tikzcd} S^{n-1} \arrow[r] \arrow[d] & X \arrow[d, "f"] \\ D^n \arrow[r] \arrow[ur, dashed, "\ell"] & Y \end{tikzcd}$$

a lift ℓ exists.

Theorem (Whitehead): A map $f: X \to Y$ between CW complexes is a homotopy equivalence $\Leftrightarrow$ it is a weak homotopy equiv.

Motivation

- **Combinatorial models** - Homotopy theory of topological spaces has algebraic structure, and seems well modeled by comb. objects (e.g. simplicial approx.) - can we make this precise?

- **Localization** - Suppose we have a category $\mathcal{C}$, $W \subseteq \mathcal{C}$ a class of maps we would like to invert to form new category $\mathcal{C}'$
- Maps $X \to Y$ in $\mathcal{C}'$ should be chains

$$X \to X_1 \text{``}\leftarrow\text{''} X_2 \to X_3 \text{``}\leftarrow\text{''} \dots \to Y$$

- Problem is to verify that this gives a <u>set</u> of maps $\mathrm{Hom}_{\mathcal{C}'}(X, Y)$

Abstract Nonsense

- $\mathcal{C}$ a category, X and Y objects, use
  $\mathrm{Hom}_{\mathcal{C}}(X,Y)$ = set of all maps $f: X \to Y$

  SET, TOP, MOD_R

- $\mathcal{C}$ and $\mathcal{D}$ categories, $F: \mathcal{C} \to \mathcal{D}$, $G: \mathcal{D} \to \mathcal{C}$
  functors are **adjoint** if
  $$\mathrm{Hom}_{\mathcal{D}}(F(X), Y) \cong \mathrm{Hom}_{\mathcal{C}}(X, G(Y)) \quad \begin{matrix} X \in \mathcal{C} \\ Y \in \mathcal{D} \end{matrix}$$

- **Limits** and **Colimits** : products, coproducts,

$$\begin{tikzcd}
X \arrow[dr, dashed] \arrow[drr, bend left] \arrow[ddr, bend right] & & \\
& L \arrow[r] \arrow[d] & B \arrow[d] \\
& A \arrow[r] & Y
\end{tikzcd}$$

(pullback)

$$\begin{tikzcd}
Y \arrow[r] \arrow[d] & A \arrow[d] \arrow[ddr, bend left] & \\
B \arrow[r] \arrow[drr, bend right] & C \arrow[dr, dashed] & \\
& & X
\end{tikzcd}$$

(pushout)

$$\begin{tikzcd}
Y_1 \arrow[r] \arrow[dr] & Y_2 \arrow[r] \arrow[d] & Y_3 \arrow[r] \arrow[dl] & \cdots \\
& Y \arrow[d, dashed] & & \\
& X & &
\end{tikzcd}$$

(seq. colimit)

- $\mathcal{C}$ and $\mathcal{D}$ categories, $\mathcal{C}^{\mathcal{D}}$ is the **functor category**, objects are functors $F: \mathcal{D} \to \mathcal{C}$

The Axioms

Defn A model category is a category $\mathcal{C}$ with 3 distinguished classes of maps

- weak equivalences $\xrightarrow{\sim}$
- fibrations $\twoheadrightarrow$
- cofibrations $\hookrightarrow$

closed under $\circ$ and contains identity maps

MC1: Finite limits and colimits exist

MC2: (2 of 3) If f, g are maps s.t. 2 out of f, g, gf is weak equiv.

MC3: If f is a retract of g, and g is a weak equiv, fib, or cofib then so is f.

MC4: Given diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed] & Y \end{tikzcd}$$

a lift exists if

- f is acyclic cofibration and g is fibration
- f is cofibration and g is acyclic fibration

MC5: Any map $f: X \to Y$ can be factored as

$$\begin{tikzcd} X \arrow[r, hook] \arrow[rr, bend right, "f"'] & X' \arrow[r, two heads, "\sim"] & Y \end{tikzcd} \qquad \begin{tikzcd} X \arrow[r, hook, "\sim"] \arrow[rr, bend right, "f"'] & X'' \arrow[r, two heads] & Y \end{tikzcd}$$

Lifting properties

<u>Defn</u> Given maps $f: A \to B$, $g: X \to Y$

say f has left lifting property (LLP) w.r.t. g
and g has right lifting property (RLP) w.r.t. f
if given any diagram

$$\begin{tikzcd} A \arrow[r] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r] \arrow[ur, dashed, "\ell"] & Y \end{tikzcd}$$

a lift ℓ exists

lemma: Suppose $\mathcal{C}$ is a model category

{Cofibrations} = {Maps with LLP w.r.t. acyclic fibrations}

{Acyclic cofibrations} = { " LLP w.r.t. fibrations }

{Fibrations} = {Maps with RLP w.r.t. acyclic cofibrations}

{Acyclic fibrations} = { " RLP w.r.t. fibrations }

$\Rightarrow$ e.g. in setting up model category, if we know weak equivs and fibrations, the cofibrations are determined.

MC1 $\Rightarrow$ $\mathcal{C}$ has both initial object $\emptyset$
and terminal object $*$

call an object $A \in \mathcal{C}$

- fibrant if $A \to *$ is a fibration
- cofibrant if $\emptyset \to A$ is a cofibration

Thm: The category TOP of top. spaces has a model category structure by defining a map $f: X \to Y$ to be

- weak equiv if $f: X \to Y$ is a weak hom. equiv.
- fibration if f is a Serre fibration
- cofibration if Y is obtained from X by "attaching cells" (or a retract of such a map)

= maps with LLP with respect to fibrations

- Duality
- Here every object is fibrant, cofibrant objects are (retracts) of generalized CW-complexes
- The homotopy category $Ho(\mathcal{C})$ is equiv. to usual homotopy category of CW-complexes

- The category CH_R of chain complexes of R-modules has objects

$$M = \quad \cdots \to M_k \xrightarrow{\partial_k} M_{k-1} \to \cdots \to M_1 \xrightarrow{\partial_1} M_0$$

where M_i is an R-module and $\partial_{i-1}\partial_i = 0$

Morphisms $M \to N$ consists of $f_i : M_i \to N_i$ s.t. $\partial f_i = f_{i-1}\partial$

- If M is a chain complex, the homology of M is $H_i(M) = \ker \partial_i / \operatorname{im} \partial_{i+1}$

Thm CH_R has a model category structure if we set $f : M \to N$ to be a

- weak equivalence if f induces isos $H_k(M) \xrightarrow{\cong} H_k(N)$
- cofibration if $f_k : M_k \to N_k$ is monomorphism with projective R-module as its cokernel
- fibration if $f_k : M_k \to N_k$ epimorphism

Homotopy

Lem: If $A, X \in \mathcal{C}$ with A cofibrant and X fibrant, then

$$\pi^R(A,X) = \pi^L(A,X) =: \pi(A,X)$$

homotopy classes of maps $A \to X$

Lem Suppose $f: A \to X$ with A, X both fibrant and cofibrant.

Then f is a weak equiv $\Leftrightarrow$ f has homotopy inverse $g: X \to A$

Pf "$\Rightarrow$" Factor f as $A \overset{p}{\underset{\sim}{\hookrightarrow}} Y \overset{q}{\twoheadrightarrow} X$ (MC5)

q is a weak equivalence (MC2)

$A \underset{p}{\hookrightarrow} Y$, A fibrant $\Rightarrow$

$$\begin{tikzcd} A \arrow[r, "id"] \arrow[d, "p"'] & A \arrow[d] \\ Y \arrow[r] \arrow[ur, dashed, "r"] & * \end{tikzcd}$$

$rp = id_A$ (MC4)

check: p induces bijection $\pi^R(Y,Y) \to \pi^R(A,Y)$

$\Rightarrow$ r is homotopy inverse for p

Dual argument gets s, homotopy inverse for q

$rs: X \to A$ is desired map.

Homotopy Category of $\mathcal{C}$

<u>Defn</u> A **cylinder object** for $A \in \mathcal{C}$ is an object $A \wedge I$ which factors the folding map:

$$\begin{tikzcd}
A \sqcup A \arrow[r, hook] \arrow[rr, bend right, "\nabla"'] & A \wedge I \arrow[r, "\sim"] & A
\end{tikzcd}$$

<u>Defn</u> $f, g : A \to X$ are **left homotopic** if there exists a map $F : A \wedge I \to X$

$$\begin{tikzcd}
A \arrow[d] \arrow[drr, "f"] & & \\
A \sqcup A \arrow[r, hook] & A \wedge I \arrow[r] & X \\
A \arrow[u] \arrow[urr, "g"'] & &
\end{tikzcd}$$

for some cylinder object $A \wedge I$.

Lemma: If A is cofibrant this is an equiv. rel.

Let $\pi^L(A, X)$ denote the set of equivalence classes

<u>Defn</u> A **path object** for $X \in \mathcal{C}$ is an object X^I which factors the diagonal map:

$$\begin{tikzcd} X \arrow[r, "\sim"] \arrow[rr, bend right, "\Delta"'] & X^I \arrow[r, two heads] & X \times X \end{tikzcd}$$

<u>Defn</u> $f, g : A \to X$ are **right homotopic** if there exists a map $H : A \to X^I$

$$\begin{tikzcd} & & & X \\ A \arrow[r] \arrow[urrr, "f"] \arrow[drrr, "g"'] & X^I \arrow[r, two heads] & X \times X \arrow[ur] \arrow[dr] & \\ & & & X \end{tikzcd}$$

for some path object X^I.

Lemma: If X is fibrant this is an equiv rel.

Let $\pi^R(A, X)$ denote the set of equivalence classes

The Homotopy Category!

- want a functor $\mathcal{C} \to Ho(\mathcal{C})$, $\mathcal{C}$ a model category
- Want fibrant and cofibrant "replacements".

factor $\emptyset \to X$ as $\emptyset \hookrightarrow QX \overset{\sim}{\twoheadrightarrow} X$

cofibrant

$X \to *$ as $X \overset{\sim}{\hookrightarrow} RX \twoheadrightarrow *$

fibrant

<u>Defn:</u> The homotopy category $Ho(\mathcal{C})$ of $\mathcal{C}$ has objects same as $\mathcal{C}$, with maps

$$\mathrm{Hom}_{Ho(\mathcal{C})}(X,Y) = \pi(RQX, RQY)$$

functor $\varphi : \mathcal{C} \to Ho(\mathcal{C})$

<u>Prop</u> If f is a map in $\mathcal{C}$, then $\varphi(f)$ is an iso. in $Ho(\mathcal{C}) \Leftrightarrow f$ is a weak equiv.

Maps in $Ho(\mathcal{C})$ generated by φ-images of maps of $\mathcal{C}$ and inverses of weak-equivs of $\mathcal{C}$

$$X \to X_1 \xleftarrow{\sim} X_2 \to X_3 \to Y$$

<u>Thm</u> $\varphi : \mathcal{C} \to Ho(\mathcal{C})$ is a localization of $\mathcal{C}$ w.r.t. weak equivs

$Ho(CH_R)$, Homotopy in CH_R

- Define a "path" $I = \ldots \to 0 \to 0 \to R \to R$
- Then if M is a chain complex, $M \times I$ is a cylinder object for M, and (left) homotopy recovers notion of chain homotopy.
- The "n-sphere": $\ldots \to 0 \to \underset{n}{R} \to 0 \to \ldots \to 0$
- Cofibrant objects = projective resolutions
- Define $K(M,n)$, for R-module M:

  $\ldots \to 0 \to \underset{n}{M} \to 0 \to \ldots \to 0$

  "Eilenberg-Maclane space"
- <u>Thm</u> For R-modules M and N

  $$\mathrm{Hom}_{Ho(CH_R)}(K(M,m), K(N,n)) \cong \mathrm{Ext}_R^{n-m}(M,N)$$

<u>Pf</u> cofibrant replacement $P \to K(M,0)$ is a projective resolution of M

maps $f,g : P \to K(N,n)$ related by right homotopy $\Leftrightarrow$ cohomological condition is met.

Combinatorial "Model"

Let Δ denote the category whose objects are the ordered sets $[n] = \{0,1,\dots,n\}$ and whose morphisms are order-preserving maps between them $i \le j \Rightarrow f(i) \le f(j)$

Δ^{op} is generated by

face maps: $d_i(0 \to 1 \to \dots \to n-1 \to n)$
$= 0 \to 1 \to \dots \to i-1 \to i+1 \to \dots \to n$

degeneracy maps: $s_i(0 \to 1 \to \dots \to n-1 \to n)$
$= 0 \to \dots \to i-1 \to i \to i \to i+1 \to \dots \to n$

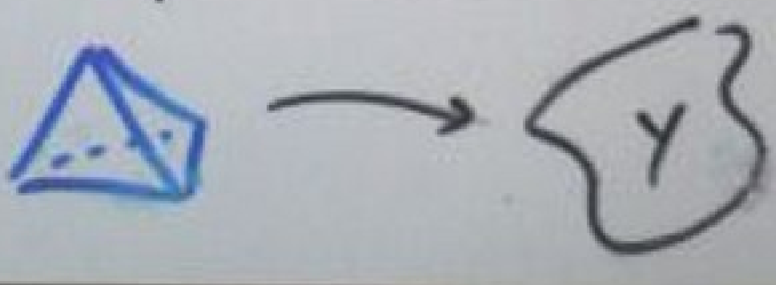
<u>Defn</u> A simplicial set X is a functor $\Delta \overset{X}{\to} SET$

The set $X([n])$ is the set of n-simplices

- Similar to simplicial complex "with singularities"

<u>Ex</u> Let Δ^n denote standard n-simplex, Y space

Form simplicial set $Sing(Y)$ by taking as n-simplices set of all cont. maps $\Delta^n \to Y$



Sing : TOP $\to$ SSET

If X is a simplicial set, then can take $|X|$, geometric realization of X by constructing a space according to "gluing" information in X

Theorem (Quillen) The following classes determine a model category structure on SSET: $f: X \to Y$ called a

- weak equivalence if $|f|: |X| \to |Y|$ is a weak equiv
- cofibration if $f_n: X([n]) \hookrightarrow Y([n])$
- fibration if f has RLP w.r.t. to acyclic cofibrations (= "Kan fibrations")

and the adjoint functors

$$| \; | : \mathrm{SSET} \rightleftarrows \mathrm{TOP} : \mathrm{Sing}$$

induces an equivalence of categories

$$\mathrm{Ho}(\mathrm{SSET}) \xrightarrow{\simeq} \mathrm{Ho}(\mathrm{TOP})$$

Further applications

Derived functors - Given functor $F: \mathcal{C} \to \mathcal{D}$ between model categories, want a functor $LF: Ho(\mathcal{C}) \to Ho(\mathcal{D})$, best possible factorization of F through $Ho(\mathcal{C})$

Homotopy (co)limits - Given model category $\mathcal{C}$ construct model category $\mathcal{C}^{\mathcal{D}}$

Simplicial objects - For many categories $\mathcal{C}$, the category $s\mathcal{C}$ carries a model structure

For the case $\mathcal{C} = MOD_R$, turns out $sMOD_R$ is equivalent to CH_R and this recovers model structure we discussed.

In general, $s\mathcal{C}$ is "**homotopical algebra**" over $\mathcal{C}$ (homotopical algebra over $\mathcal{C}$ <u>is</u> ordinary homotopy theory)

Rational homotopy theory, ...